

# On the dynamical degrees of meromorphic maps preserving a fibration

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August 25, 2011

## Abstract

Let  $f$  be a dominant meromorphic self-map on a compact Kähler manifold  $X$  which preserves a meromorphic fibration  $\pi : X \rightarrow Y$  of  $X$  over a compact Kähler manifold  $Y$ . We compute the dynamical degrees of  $f$  in term of its dynamical degrees relative to the fibration and the dynamical degrees of the map  $g : Y \rightarrow Y$  induced by  $f$ . We derive from this result new properties of some fibrations intrinsically associated to  $X$  when this manifold admits an interesting dynamical system.

**Classification AMS 2010:** 37F, 14D (primary), 32U40, 32H50 (secondary).

**Keywords:** semi-conjugate maps, dynamical degree, relative dynamical degree.

## 1 Introduction

Let  $X$  be a compact Kähler manifold of dimension  $k$  and  $\omega_X$  a Kähler form on  $X$ . Consider a meromorphic self-map  $f : X \rightarrow X$ . Assume that  $f$  is *dominant*, i.e. the image of  $f$  contains an open subset of  $X$ . The iterate of order  $n$  of  $f$  is defined by  $f^n := f \circ \cdots \circ f$  ( $n$  times) on a dense Zariski open set and extends to a dominant meromorphic map on  $X$ .

Define, for  $0 \leq p \leq k$  and  $n \geq 0$ ,

$$\lambda_p(f^n) := \|(f^n)^*(\omega_X^p)\| = \int_X (f^n)^*(\omega_X^p) \wedge \omega_X^{k-p}.$$

It was shown in [9, 10] that the sequence  $[\lambda_p(f^n)]^{1/n}$  converges to a constant  $d_p(f)$  which is the *dynamical degree of order  $p$*  of  $f$ . It measures the growth of the norms of  $(f^n)^*$  acting on the Hodge cohomology group  $H^{p,p}(X, \mathbb{R})$  when  $n$  tends to infinity.

Dynamical degrees  $d_p(f)$  play a central role in the study of the dynamical system associated to  $f$ , e.g. on the computation of entropies, the construction

of invariant currents and the equidistribution problems. We refer the reader to [12, 14, 18, 24] for more results on this matter.

By the mixed version of Hodge-Riemann bilinear relations [7, 13, 16, 20, 21], the dynamical degrees of  $f$  are log-concave, i.e.  $p \mapsto \log d_p(f)$  is concave or equivalently  $d_p(f)^2 \geq d_{p-1}(f)d_{p+1}(f)$  for  $1 \leq p \leq k-1$ . Therefore, there are integers  $p \leq p'$  such that

$$1 = d_0(f) < \cdots < d_p(f) = \cdots = d_{p'}(f) > \cdots > d_k(f) \geq 1.$$

An important problem in Complex Dynamics is to find dynamically interesting examples of meromorphic self-maps on compact Kähler manifolds. We may rephrase the question in a different way by characterizing manifolds  $X$  on which there is a self-map  $f$  with distinct consecutive dynamical degrees, i.e. with  $p = p'$ , since this condition prevents the associated dynamical system from containing neutral directions, e.g.  $f = \text{id}_Y \times g$  on  $X = Y \times Z$  for some meromorphic self-map  $g$  on  $Z$ .

A meromorphic self-map  $f : X \rightarrow X$  always preserves certain natural meromorphic fibrations associated to  $X$  that we encounter in Algebraic Geometry, see e.g. Amerik-Campana [1] and Nakayama-Zhang [17, 26]. These fibrations are the key tool in the classification theory of algebraic varieties and compact complex spaces, see e.g. Ueno's book [22]. So, in order to answer the above question we are led, in a natural way, to study self-maps which preserve fibrations.

Let  $\pi : X \rightarrow Y$  be a dominant meromorphic map from  $X$  onto a compact Kähler manifold  $Y$  of dimension  $l \leq k$ . This map defines a fibration on  $X$  which might be singular. Suppose that  $f$  preserves this fibration, i.e.  $f$  sends generic fibers of  $\pi$  to fibers of  $\pi$ . This property is equivalent to the existence of a dominant meromorphic map  $g : Y \rightarrow Y$  such that  $\pi \circ f = g \circ \pi$ . We say that  $f$  is *semi-conjugate* to  $g$  or more precisely,  $\pi$ -*semi-conjugate* to  $g$ . In this context, the first and second authors have introduced in [8] the *dynamical degree*  $d_p(f|\pi)$  of order  $p$  of  $f$  relative to  $\pi$  for  $0 \leq p \leq k-l$ . Roughly speaking, this quantity measures the growth of  $(f^n)^*$  acting on the subspace  $H_\pi^{l+p, l+p}(X, \mathbb{R})$  of classes in  $H^{l+p, l+p}(X, \mathbb{R})$  which can be supported by a generic fiber of  $\pi$ . Precise formulations will be recalled in Section 3 below.

The main purpose of the present work is to quantify the relation between the dynamical systems associated to semi-conjugate maps. In our view, this quantification, which is formulated in terms of dynamical degrees, has at least two immediate consequences. First, it will shed a light to the above question of characterizing manifolds with interesting dynamical systems. Second, it shows that the dynamical system of a map  $f$  could be understood by studying the dynamics of a simpler map  $g$  to which  $f$  is semi-conjugate. Here is our main result.

**Theorem 1.1.** *Let  $X$  and  $Y$  be compact Kähler manifolds of dimension  $k$  and  $l$  respectively with  $k \geq l$ . Let  $f : X \rightarrow X$ ,  $g : Y \rightarrow Y$  and  $\pi : X \rightarrow Y$  be dominant*

meromorphic maps such that  $\pi \circ f = g \circ \pi$ . Then, we have

$$d_p(f) = \max_{\max\{0, p-k+l\} \leq j \leq \min\{p, l\}} d_j(g) d_{p-j}(f|_{\pi})$$

for every  $0 \leq p \leq k$ .

Theorem 1.1 completes the work in [8] where the case when  $X$  and  $Y$  are projective manifolds has been proved. Note that the condition  $\max\{0, p-k+l\} \leq j \leq \min\{p, l\}$  is equivalent to  $0 \leq j \leq l$  and  $0 \leq p-j \leq k-l$  which insures that  $d_j(g)$  and  $d_{p-j}(f|_{\pi})$  are meaningful. We have the following useful consequence.

**Corollary 1.2.** *Let  $f, \pi, g$  be as in Theorem 1.1. If the consecutive dynamical degrees of  $f$  are distinct, then the same property holds for  $g$  and for the consecutive dynamical degrees of  $f$  relative to  $\pi$ .*

We deduce from Corollary 1.2 various algebro-geometric properties of manifolds admitting dynamically interesting self-maps. The following result is obtained using the Iitaka fibrations of  $X$ .

**Corollary 1.3.** *Let  $X$  be a compact Kähler manifold admitting a dominant meromorphic self-map with distinct consecutive dynamical degrees. Then, the Kodaira dimension of  $X$  is either equal to 0 or  $-\infty$ .*

Note that the same result was proved for compact Kähler surfaces by Cantat in [5] and Guedj in [15], for holomorphic maps on compact Kähler manifolds by Nakayama and Zhang in [17, 25], and for meromorphic maps on projective manifolds by the first and second authors in [8].

Applying Corollary 1.2 to the Albanese fibration of  $X$ , we get the following result.

**Corollary 1.4.** *Let  $X$  be a compact Kähler manifold admitting a dominant meromorphic self-map with distinct consecutive dynamical degrees. Then, the Albanese map  $\text{alb} : X \rightarrow \text{Alb}(X)$  is surjective.*

In his recent works [4], Campana has constructed, for an arbitrary compact Kähler manifold  $X$ , its core fibration. This fibration functorially decomposes  $X$  into its special components (the fibers) and its general type component (orbifold base). In the light of his construction, we obtain the following result as a consequence of Corollary 1.2.

**Corollary 1.5.** *Let  $X$  be a compact Kähler manifold admitting a dominant meromorphic self-map with distinct consecutive dynamical degrees. Then,  $X$  is special in the sense of Campana, see Definition 2.1 in [4] for the terminology.*

We give here the outlines of the article. The main ingredient in the proof of Theorem 1.1 is the introduction of an approximation calculus for positive closed currents on compact Kähler manifolds which is well-adapted with respect to a holomorphic (possibly singular) fibration. Using the peculiarity of projective manifolds a primitive form of this calculus has been achieved in [8] where it has played a key role in the proof of the main result, see Proposition 2.3 and Proposition 2.4 therein.

This peculiarity is not available any more in the context of general compact Kähler manifolds. Our new idea here is to reduce the above calculus, via a well-chosen bi-meromorphic model, to the problem of approximating, with mass control, positive closed currents defined on submanifolds. This is the content of Section 2 below. Section 3 is devoted to the proof of the main theorem and its corollaries. Although we adopt here the strategy of [8], our exposition is somewhat simpler and more constructive.

**Acknowledgment.** The third author would like to thank Professor Eric Bedford for introducing the work in the paper [8], which initiated the interest in this project. He also would like to thank Professors János Kollár, Jaroslaw Włodarczyk, Valery Lunts and Sergey Pinchuk for helps. This paper was partially written during the visit of the first author at Humboldt Universität zu Berlin and the visit of the third author at University Paris-Sud. They would like to thank these organizations, the Alexander von Humboldt foundation, Professors Jürgen Leiterer and Nessim Sibony for their supports and their hospitality.

## 2 Calculus on positive closed currents

In this section, we develop a delicate approximation theory for positive closed currents on compact Kähler manifolds. This is the key ingredient for our method. Note that by positive currents we mean positive currents in the strong sense, see e.g. [12, A.2] for the terminology. We will mostly apply our results to either currents of integration on varieties or *almost-smooth* currents, i.e. currents given by  $L^1$ -forms which are smooth outside a proper analytic subset. We refer the reader to the books by Demailly [6] and Voisin [23] for the basic facts on currents and on Kähler geometry.

In what follows, if  $T$  is a current and  $\phi$  is a differential form on a manifold  $M$ , both the pairings  $\langle T, \phi \rangle$  and  $\langle \phi, T \rangle$  denote the value of  $T$  at  $\phi$ . In particular, when  $T$  is also a differential form, these pairings are equal to the integral of  $T \wedge \phi$  on  $M$ . The cohomology class of a closed current is denoted by  $\{\cdot\}$  and the current of integration on an analytic set is denoted by  $[\cdot]$ .

Consider a compact Kähler manifold  $(X, \omega_X)$  of dimension  $k$  as above. Let  $\mathcal{K}^p(X)$  denote the cone of classes of smooth strictly positive closed  $(p, p)$ -forms in  $H^{p,p}(X, \mathbb{R})$ . This is an open salient cone, i.e.  $\overline{\mathcal{K}^p(X)} \cap -\overline{\mathcal{K}^p(X)} = \{0\}$ . If

$c, c'$  are two classes in  $H^{p,p}(X, \mathbb{R})$ , we write  $c \leq c'$  and  $c' \geq c$  when  $c' - c$  is a class in  $\overline{\mathcal{K}^p(X)}$ .

If  $T, T'$  are two real currents of bidegree  $(p, p)$ , we write  $T \geq T'$  and  $T' \leq T$  when  $T - T'$  is a positive current. If  $T$  is a positive closed  $(p, p)$ -current, the *mass* of  $T$  is defined by  $\|T\| := \langle T, \omega_X^{k-p} \rangle$ . This quantity is equivalent to the classical mass-norm of  $T$  but it has the advantage that it depends only on the cohomology class  $\{T\}$  of  $T$ .

The following semi-regularization of currents was proved by Sibony and the first author in [9, 10].

**Proposition 2.1.** *Let  $T$  be a positive closed  $(p, p)$ -current on a compact Kähler manifold  $(X, \omega_X)$ . Then, there is a sequence of smooth positive closed  $(p, p)$ -forms  $T_n$  on  $X$  which converges weakly to a positive closed  $(p, p)$ -current  $T' \geq T$  such that  $\|T_n\| \leq A\|T\|$  and  $\{T_n\} \leq A\|T\|\{\omega_X^p\}$ , where  $A > 0$  is a constant that depends only on  $(X, \omega_X)$ . In particular, we have  $\{T\} \leq A\|T\|\{\omega_X^p\}$ . Moreover, if  $T$  is smooth on an open set  $U$ , then for every compact set  $K \subset U$ , we have  $T_n \geq T$  on  $K$  when  $n$  is large enough.*

This semi-regularization of currents is the main technical tool in the proof of several results in Complex Dynamics. However, for the main results of this work, we will need refined versions of the above proposition which somehow take into account the presence of a fibration on  $X$ . We will give now some statements in an abstract setting which may have an independent interest. The following result generalizes Proposition 2.1.

**Proposition 2.2.** *Let  $(X, \omega_X)$  be a compact Kähler manifold of dimension  $k$  and  $W$  a submanifold of dimension  $r$  of  $X$ . Let  $\iota : W \hookrightarrow X$  denote the canonical embedding. Let  $S$  be a positive closed  $(p, p)$ -current of mass 1 on  $W$ . Then, there are smooth positive closed  $(p, p)$ -currents  $T_n$  on  $X$  such that  $\iota^*(T_n)$  converge to a current  $S' \geq S$  and that the masses of  $T_n$  are bounded by a constant  $c = c(X, \omega_X, W)$  which is independent of  $S$ .*

*Proof.* By Proposition 2.1 applied to the current  $S$  on  $W$ , it is enough to consider the case where  $S$  is smooth, see also Lemma 2.3 below. Let  $\pi_1, \pi_2$  denote the canonical projections from  $W \times X$  onto its factors. The idea of the proof is to write

$$\iota_*(S) = (\pi_2)_*(\pi_1^*(S) \wedge [\Delta_W]).$$

Then, in order to obtain  $T_n$ , we have just to replace  $[\Delta_W]$  by a suitable smooth positive closed current on  $W \times X$ .

Let  $\Pi : \widehat{W \times X} \rightarrow W \times X$  be the blow-up of  $W \times X$  along the diagonal  $\Delta_W$  of  $W \times W$ . By Blanchard's theorem [3],  $\widehat{W \times X}$  is a Kähler manifold. Fix a large enough Kähler form  $\omega$  on  $\widehat{W \times X}$ . Consider  $U := \Pi_*(\omega)$ . Then,  $U$  is a positive closed  $(1, 1)$ -current on  $W \times X$  which is smooth outside the diagonal  $\Delta_W$  and

have Lelong number  $\geq 1$  at each point of  $\Delta_W$ . Adding to  $\omega$  the pull-back of a Kähler form on  $W \times X$  allows to assume that  $U$  belongs to a Kähler class.

Fix a Kähler form  $\alpha$  in the cohomology class  $\{U\}$ . There is a quasi-p.s.h. function  $u$  on  $W \times X$  such that  $U = \alpha + dd^c u$ . Such a function is called a *quasi-potential* of  $U$ . We claim that there is a sequence of smooth positive closed  $(1, 1)$ -forms  $U_n$  with decreasing smooth quasi-potentials  $u_n$  such that  $\lim_{n \rightarrow \infty} U_n = U$ , that is,  $U_n = \alpha + dd^c u_n$  and  $u_n \searrow u$  as  $n \nearrow \infty$ . Indeed, it is enough to take  $U_n := \alpha + dd^c u_n$ , where we define  $u_n$  as  $\max_{\epsilon_n}(u, -n)$  for a suitable regularization  $\max_{\epsilon_n}(\cdot, \cdot)$  of the function  $\max(\cdot, \cdot)$ .

Define

$$T_n := (\pi_2)_*(\pi_1^*(S) \wedge U_n^r).$$

Since  $U_n$  is smooth and the  $\pi_i$  are submersions,  $T_n$  is also smooth. All currents we consider have masses bounded by a constant  $c = c(X, \omega_X, W)$  because their cohomology classes are bounded. Extracting a subsequence allows us to assume that  $\iota^*(T_n)$  converge to a current  $S'$ . To complete the proof it suffices to check that  $S' \geq S$ .

Let  $\Phi$  be an arbitrary *weakly* positive test form of bidegree  $(r-p, r-p)$  on  $W$ , see e.g. [12, A.2] for the terminology. We need to check that  $\langle S', \Phi \rangle \geq \langle S, \Phi \rangle$ . Let  $\tau_1, \tau_2$  denote the canonical projections from  $W \times W$  onto its factors. Consider the diagram

$$W \times W \xrightarrow{\text{id}_W \times \iota} W \times X.$$

We have for  $S$  smooth

$$\begin{aligned} \langle \iota^*(T_n), \Phi \rangle &= \langle (\pi_2)_*(\pi_1^*(S) \wedge U_n^r), \iota_* \Phi \rangle \\ &= \langle S, (\pi_1)_*(U_n^r \wedge \pi_2^* \iota_* \Phi) \rangle \\ &= \langle S, (\tau_1)_*((\text{id}_W \times \iota)^* U_n^r \wedge \tau_2^* \Phi) \rangle. \end{aligned}$$

Note that the above identities hold also for smooth currents  $S$  which are not positive closed. Moreover, the first and the last integrals are meaningful for all  $S$  of order 0 and depend continuously on  $S$ . Thus, by continuity, these integrals are also equal for  $S$  as in our proposition. It follows that

$$\langle S', \Phi \rangle = \lim_{n \rightarrow \infty} \langle \iota^*(T_n), \Phi \rangle = \lim_{n \rightarrow \infty} \langle S, (\tau_1)_*((\text{id}_W \times \iota)^* U_n^r \wedge \tau_2^* \Phi) \rangle.$$

Therefore, in order to show that the last integral is greater than  $\langle S, \Phi \rangle$  it suffices to check that any limit value of the sequence  $(\text{id}_W \times \iota)^* U_n^r$  is larger than or equal to  $[\Delta_W]$ .

To obtain this inequality, we suppose without loss of generality that the sequence  $(\text{id}_W \times \iota)^* U_n^r$  converges weakly to a current  $U'$ . Clearly,

$$\begin{aligned} (\text{id}_W \times \iota)_* U' &= \lim_{n \rightarrow \infty} (\text{id}_W \times \iota)_*(\text{id}_W \times \iota)^* U_n^r \\ &= \lim_{n \rightarrow \infty} U_n^r \wedge [W \times W] \\ &= U^r \wedge [W \times W], \end{aligned}$$

because the  $U_n$  admit quasi-potentials which decrease to a quasi-potential of  $U$ . Note that the last wedge-product is well-defined since  $U$  is smooth outside  $\Delta_W$  and  $\dim \Delta_W = r$ , see e.g. Demailly [6].

Hence, we only need to show that  $U^r \wedge [W \times W] \geq [\Delta_W]$ . But this can be checked using a local model of the blow-up  $\Pi : \widehat{W \times X} \rightarrow W \times X$ . Indeed, consider a  $(k+r)$ -dimensional polydisc  $\mathbb{D}$  in  $W \times X$  with holomorphic coordinates  $z_1, \dots, z_{k+r}$  around an arbitrary fixed point in  $W \times W$ . We can choose these local coordinates so that  $W \times W$  is equal to the linear subspace  $\{z_{r+1} = \dots = z_k = 0\}$  and  $\Delta_W$  is equal to the linear subspace  $\{z_1 = \dots = z_k = 0\}$ . Let  $[w_1 : \dots : w_k]$  be the homogeneous coordinates on  $\mathbb{P}^{k-1}$ . Then,  $\widehat{W \times X} \cap \Pi^{-1}(\mathbb{D})$  may be identified with the smooth manifold

$$\widehat{\mathbb{D}} := \{(z_1, \dots, z_{k+r}, [w_1 : \dots : w_k]) \in \mathbb{D} \times \mathbb{P}^{k-1} : z_i w_j = z_j w_i \text{ for } 1 \leq i, j \leq k\}.$$

Observe that  $\Pi$  is induced by the canonical projection from  $\mathbb{D} \times \mathbb{P}^{k-1}$  onto the factor  $\mathbb{D}$ . Let  $\Pi'$  be the canonical projection from  $\widehat{\mathbb{D}}$  onto the factor  $\mathbb{P}^{k-1}$ . Let  $\omega_{\text{FS}}$  be the Fubini-Study form on  $\mathbb{P}^{k-1}$ . Recall that  $\omega_{\text{FS}}$  is induced by the  $(1,1)$ -form  $dd^c \log \|(w_1, \dots, w_k)\|$  on  $\mathbb{C}^k \setminus \{0\}$ . Since  $\omega$  is large enough, we have  $\omega \geq \Pi'^*(\omega_{\text{FS}})$ . Since  $[w_1 : \dots : w_k] = [z_1 : \dots : z_k]$  outside the exceptional hypersurface  $\Pi^{-1}(\Delta_W)$  of  $\widehat{D}$ , we obtain

$$\Pi_*(\omega) \geq \Pi_* \Pi'^*(\omega_{\text{FS}}) = dd^c \log \|(z_1, \dots, z_k)\|.$$

This inequality holds on  $\mathbb{D} \setminus \Delta_W$  and hence on  $\mathbb{D}$  since positive closed  $(1,1)$ -currents have no mass on subvarieties of codimension  $\geq 2$ .

Finally, we deduce that

$$\begin{aligned} U^r \wedge [W \times W] &= \Pi_*(\omega)^r \wedge [W \times W] \\ &\geq (dd^c \log \|(z_1, \dots, z_k)\|)^r \wedge [W \times W] \\ &= (dd^c \log \|(z_1, \dots, z_r)\|)^r \wedge [W \times W]. \end{aligned}$$

The last current is equal to  $[\Delta_W]$ . So, the proof of the lemma is completed.  $\square$

Before giving the main result in this section, let us introduce some useful notions that we will need in our computation. Let  $(M, \omega_M)$  be a compact Kähler manifold of dimension  $m$ . In [11] Sibony and the first author have introduced the following natural metric on the space of positive closed  $(p,p)$ -currents on  $M$ . If  $R$  and  $S$  are such currents, define

$$\text{dist}(R, S) := \sup_{\|\Phi\|_{\mathcal{C}^1} \leq 1} |\langle R - S, \Phi \rangle|,$$

where  $\Phi$  is a smooth  $(m-p, m-p)$ -form on  $M$  and we use the sum of  $\mathcal{C}^1$ -norms of its coefficients for a fixed atlas on  $M$ . Recall the following result from Proposition 2.1.4 in [11].

**Lemma 2.3.** *On the convex set of positive closed  $(p, p)$ -currents of mass  $\leq 1$  on  $M$ , the topology induced by the above distance coincides with the weak topology.*

Consider now a dominant meromorphic map  $h : (M, \omega_M) \rightarrow (N, \omega_N)$  between compact Kähler manifolds. It is well-known (see e.g. [8, 9, 10]) that  $h$  induces the linear operators  $h^*$  and  $h_*$  acting on smooth forms. In general, the above operators do not extend continuously to positive closed currents. We will use instead the strict pull-back of currents  $h^\bullet$  which coincides with  $h^*$  on smooth positive closed forms.

Let  $U$  be the maximal Zariski open set in  $M$  such that  $h : U \rightarrow h(U)$  is locally a submersion. The complement of  $U$  in  $M$  is called the *critical set* of  $h$ . If  $T$  is a positive closed  $(p, p)$ -current on  $N$ ,  $(h|_U)^*(T)$  is well-defined and is a positive closed  $(p, p)$ -current on  $U$ . Proposition 2.1 allows us to show that this current has finite mass. By Skoda's theorem [19], its trivial extension to  $M$  is a positive closed  $(p, p)$ -current that we denote by  $h^\bullet(T)$ . We will use the property that

$$\|h^\bullet(T)\| \leq A\|T\| \quad (1)$$

for some constant  $A > 0$  independent of  $T$ , see [9, 10] for details.

Let  $T$  and  $S$  be positive closed currents on  $M$  of bidegrees  $(p, p)$  and  $(q, q)$  respectively with  $p + q \leq m$ . Assume that  $T$  is smooth on a dense Zariski open set  $U$  of  $M$ . Then,  $T|_U \wedge S|_U$  is well-defined and has a finite mass. Therefore, by Skoda's theorem [19], its trivial extension defines a positive closed current on  $M$ . We denote by  $T \overset{\circ}{\wedge} S$  this current obtained for the maximal Zariski open set  $U$  on which  $T$  is smooth (in that case  $T|_U$  is the regular part of  $T$ ). Observe that when  $S$  has no mass on proper analytic subsets of  $M$ , the current obtained in this way does not change if we replace  $U$  with a smaller dense Zariski open set. We have the following result, see Lemma 2.2 in [8].

**Lemma 2.4.** *There is a constant  $A > 0$  independent of  $T$  and  $S$  such that*

$$\|T \overset{\circ}{\wedge} S\| \leq A\|T\|\|S\|.$$

We now state the main result of this section. It is the key technical tool in our proof of the main theorem. Let  $\pi : (X, \omega_X) \rightarrow (Y, \omega_Y)$  be a dominant holomorphic map between compact Kähler manifolds of dimension  $k$  and  $l$  respectively with  $k \geq l$ . Let  $T$  be a positive closed  $(p, p)$ -current on  $X$ . Define for  $\max\{0, p-k+l\} \leq j \leq \min\{l, p\}$ , or equivalently, for  $0 \leq j \leq l$  and  $0 \leq p-j \leq k-l$ ,

$$\alpha_j(T) := \langle T, \pi^*(\omega_Y^{l-j}) \wedge \omega_X^{k-l-p+j} \rangle. \quad (2)$$

Observe that  $\alpha_j(T)$  depends only on the cohomology class  $\{T\}$  of  $T$ . Moreover, if  $A$  is a constant such that  $\pi^*(\omega_Y) \leq A\omega_X$ , then  $\alpha_j(T) \leq A\alpha_{j+1}(T)$ .

Denote by  $\smile$  the cup-product on the Hodge cohomology ring. The following result holds for a larger class of currents  $T$  but for simplicity we limit ourself in the case that we need.



**Proposition 2.5.** *Let  $T$  be an almost-smooth positive closed  $(p, p)$ -current on  $X$ . Then, there are positive closed smooth  $(p, p)$ -forms  $T_n$  on  $X$  converging to a positive closed current  $T' \geq T$  such that*

$$\{T_n\} \leq A \sum_{\max\{0, p-k+l\} \leq j \leq \min\{l, p\}} \alpha_j(T) \{\pi^*(\omega_Y^j)\} \smile \{\omega_X^{p-j}\},$$

where  $A > 0$  is a constant that depends only on  $(X, \omega_X)$ . In particular, we have

$$\{T\} \leq A \sum_{\max\{0, p-k+l\} \leq j \leq \min\{l, p\}} \alpha_j(T) \{\pi^*(\omega_Y^j)\} \smile \{\omega_X^{p-j}\}$$

and

$$\alpha_j(T_n) \leq A \alpha_j(T)$$

for some constant  $A > 0$  that depends only on  $(X, \omega_X)$ .

Recall that  $\alpha_j(\cdot)$  is bounded by a constant times  $\alpha_{j+1}(\cdot)$ . We also have  $\omega_Y^{l+1} = 0$  since  $\dim Y = l$ . So, from the definition of  $\alpha_j(\cdot)$ , it is not difficult to see that the last assertion of Proposition 2.5 is a direct consequence of the first one. The rest of this section is devoted to the proof of the first assertion of that proposition. For this purpose we need some preparatory results.

Let  $\pi_1, \pi_2 : X \times X \rightarrow X$  be the canonical projections onto the first and second factors. Denote by  $\Delta_X$  and  $\Delta_Y$  the diagonals of  $X \times X$  and of  $Y \times Y$  respectively. Then,  $(\pi \times \pi)^{-1}(\Delta_Y)$  is an analytic subvariety of  $X \times X$  which contains  $\Delta_X$ .

**Lemma 2.6.** *There is a unique irreducible component  $V$  of  $(\pi \times \pi)^{-1}(\Delta_Y)$  which contains  $\Delta_X$ . Moreover,  $V$  has dimension  $2k - l$  and the singular locus of  $V$  does not contain  $\Delta_X$ .*

*Proof.* Let  $Z$  denote the set of critical values of  $\pi$ . By Bertini-Sard theorem,  $Z$  is a proper analytic subset of  $Y$ . Define  $Y' := Y \setminus Z$ ,  $X' := X \setminus \pi^{-1}(Z)$  and  $\pi' := \pi|_{X'}$ . So,  $\pi' : X' \rightarrow Y'$  is a submersion and  $(\pi' \times \pi')^{-1}(\Delta_{Y'})$  is a smooth complex submanifold of dimension  $2k - l$  of  $X' \times X'$ . Note that  $(\pi' \times \pi')^{-1}(\Delta_{Y'})$  is the trace of  $(\pi \times \pi)^{-1}(\Delta_Y)$  in  $X' \times X'$ . Hence, the regular part of  $(\pi \times \pi)^{-1}(\Delta_Y)$  contains  $(\pi' \times \pi')^{-1}(\Delta_{Y'})$ .

Since  $\Delta_{X'}$  is irreducible and is contained in  $(\pi' \times \pi')^{-1}(\Delta_{Y'})$  which is a smooth manifold, there is a unique irreducible component  $V'$  of that manifold which contains  $\Delta_{X'}$ . Finally, since  $\Delta_{X'}$  is a dense Zariski open set of  $\Delta_X$ , the unique irreducible component  $V$  of  $(\pi \times \pi)^{-1}(\Delta_Y)$  containing  $V'$  is also the unique component which contains  $\Delta_X$ . Its dimension is equal to  $\dim V' = 2k - l$ . Its singular locus does not contain  $\Delta_X$  since its regular part contains  $\Delta_{X'}$ .  $\square$

By Lemma 2.6 and the embedded resolution theorem of Hironaka (see [2] or Theorem 2.0.2 in [27]), there is a finite composition of blow-ups along smooth centers  $\sigma : \widetilde{X \times X} \rightarrow X \times X$  with the following properties:

- If  $E$  is the exceptional divisor of  $\sigma$ , then  $\sigma(E)$  is contained in the singular locus of  $V$ , and thus is contained in  $(X \times \pi^{-1}(Z)) \cup (\pi^{-1}(Z) \times X)$ .
- The strict transform  $\widetilde{V}$  of  $V$  is smooth, and hence is a compact Kähler submanifold of  $\widetilde{X \times X}$ . We denote by  $\iota : \widetilde{V} \hookrightarrow \widetilde{X \times X}$  the inclusion.

Because  $\Delta_X$  is not contained in  $(X \times \pi^{-1}(Z)) \cup (\pi^{-1}(Z) \times X)$ , its strict transform  $\widetilde{\Delta}_X$  is well-defined. This is an irreducible subvariety of dimension  $k$  of  $\widetilde{V}$  and its image by  $\sigma$  is  $\Delta_X$ . Since  $\Delta_X$  is also irreducible, we get that

$$\sigma_*([\widetilde{\Delta}_X]) = [\Delta_X].$$

**Lemma 2.7.** *There are smooth positive closed  $(l, l)$ -forms  $\Delta_{Y,n}$  on  $Y \times Y$  and smooth positive closed  $(k-l, k-l)$ -forms  $\Delta_{X,n}$  on  $X \times X$ , all with uniformly bounded masses, such that the limit  $\Theta := \lim_{n \rightarrow \infty} (\pi \times \pi)^*(\Delta_{Y,n}) \wedge \Delta_{X,n}$  exists and is larger or equal to  $[\Delta_X]$ .*

*Proof.* Denote by  $\Omega$  the  $(k-l, k-l)$ -current on  $\widetilde{V}$  defined as the integration on  $\widetilde{\Delta}_X$ . Then,  $\iota_*(\Omega) = [\widetilde{\Delta}_X]$  as currents on  $\widetilde{X \times X}$ . We apply Proposition 2.2 for  $\widetilde{X \times X}$  instead of  $X$ , for  $W := \widetilde{V}$ ,  $S := \Omega$  and  $p := k-l$ . Consequently, there is a sequence of smooth positive closed  $(k-l, k-l)$ -forms  $\Omega_r$  on  $\widetilde{X \times X}$  with  $\|\Omega_r\|$  uniformly bounded such that  $\lim_{r \rightarrow \infty} \iota^*(\Omega_r) \geq \Omega$ . Hence,

$$\iota_*(\Omega) \leq \lim_{r \rightarrow \infty} [\widetilde{V}] \wedge \Omega_r.$$

From the definition of  $V$ , we have  $[\widetilde{V}] \leq ((\pi \times \pi) \circ \sigma)^*[\Delta_Y]$ . By Proposition 2.1, there are smooth positive closed  $(l, l)$ -forms  $\Delta_{s,Y}$  on  $Y \times Y$  with uniformly bounded masses such that  $\lim_{s \rightarrow \infty} \Delta_{s,Y} \geq [\Delta_Y]$ . Hence,

$$[\widetilde{V}] \leq \lim_{s \rightarrow \infty} ((\pi \times \pi) \circ \sigma)^*(\Delta_{s,Y}).$$

Here, in order to get the existence of the above limit, we extract a subsequence if necessary. It follows that

$$\iota_*(\Omega) \leq \lim_{r \rightarrow \infty} \lim_{s \rightarrow \infty} ((\pi \times \pi) \circ \sigma)^*(\Delta_{s,Y}) \wedge \Omega_r.$$

Since  $\sigma_*(\iota_*(\Omega)) = [\Delta_X]$ , applying the projection formula gives

$$[\Delta_X] \leq \lim_{r \rightarrow \infty} \lim_{s \rightarrow \infty} (\pi \times \pi)^*(\Delta_{s,Y}) \wedge \sigma_*(\Omega_r).$$

Recall that  $\|\Omega_r\|$  is bounded uniformly on  $r$ . Therefore, by Proposition 2.1, there are smooth positive closed  $(k-l, k-l)$ -forms  $\Delta_{r,t,X}$  on  $X \times X$  with uniformly

bounded masses such that  $\lim_{t \rightarrow \infty} \Delta_{r,t,X} \geq \sigma_*(\Omega_r)$ . Putting the above inequalities together, we obtain

$$\lim_{r \rightarrow \infty} \lim_{s \rightarrow \infty} \lim_{t \rightarrow \infty} (\pi \times \pi)^*(\Delta_{s,Y}) \wedge \Delta_{r,t,X} \geq [\Delta_X].$$

Since  $\|\Delta_{s,Y}\|$ ,  $\|\Delta_{r,t,X}\|$  are uniformly bounded, by Lemma 2.3, we can extract two sequences  $\Delta_{X,n} := \Delta_{r_n,t_n,X}$  and  $\Delta_{Y,n} := \Delta_{s_n,Y}$  such that

$$\lim_{n \rightarrow \infty} (\pi \times \pi)^*(\Delta_{Y,n}) \wedge \Delta_{X,n} \geq [\Delta_X].$$

This completes the proof.  $\square$

**End of the proof of Proposition 2.5.** Let  $\Delta_{Y,n}$  and  $\Delta_{X,n}$  be smooth positive closed forms given by Lemma 2.7. Define

$$T_n := (\pi_1)_* [(\pi \times \pi)^*(\Delta_{Y,n}) \wedge \Delta_{X,n} \wedge \pi_2^*(T)].$$

So, the  $T_n$  are smooth positive closed  $(p,p)$ -forms on  $X$  with uniformly bounded masses. Hence, by extracting a subsequence if necessary, we can assume without loss of generality that the limit  $T' := \lim_{n \rightarrow \infty} T_n$  exists.

Let  $C$  be a proper analytic subset of  $X$  so that  $T$  is smooth on  $X \setminus C$ . Define  $U := \pi_2^{-1}(X \setminus C)$ . By Lemma 2.7, since  $T$  is smooth outside  $C$ , we get easily that

$$[\Delta_X]_{|U} \wedge \pi_2^*(T)_{|U} \leq \lim_{n \rightarrow \infty} [(\pi \times \pi)^*(\Delta_{Y,n}) \wedge \Delta_{X,n} \wedge \pi_2^*(T)]_{|U}.$$

It follows that

$$T' = \lim_{n \rightarrow \infty} T_n \geq (\pi_1)_* ([\Delta_X]_{|U} \wedge \pi_2^*(T)_{|U}) = T$$

since the last current is almost-smooth and hence has no mass on proper analytic subsets of  $X$ .

Now, we turn to the proof of the first inequality in the proposition. Let  $\tau_1, \tau_2$  denote the projections from  $Y \times Y$  onto its factors. Define two Kähler forms on  $Y \times Y$  and  $X \times X$  by

$$\omega_{Y \times Y} := \tau_1^*(\omega_Y) + \tau_2^*(\omega_Y) \quad \text{and} \quad \omega_{X \times X} := \pi_1^*(\omega_X) + \pi_2^*(\omega_X).$$

Since  $\|\Delta_{Y,n}\|$  and  $\|\Delta_{X,n}\|$  are uniformly bounded, by Proposition 2.1, there is a constant  $A_1 > 0$  independent of  $n$  so that

$$\{\Delta_{Y,n}\} \leq A_1 \{\omega_{Y \times Y}^l\} \quad \text{and} \quad \{\Delta_{X,n}\} \leq A_1 \{\omega_{X \times X}^{k-l}\}.$$

Hence, we obtain

$$\{(\pi \times \pi)^*(\Delta_{Y,n})\} \leq A_1 \{(\pi \times \pi)^*(\omega_{Y \times Y}^l)\} = A_1 \{(\pi_1^* \pi^* \omega_Y + \pi_2^* \pi^* \omega_Y)^l\}$$

and

$$\{\Delta_{X,n}\} \leq A_1 \{\omega_{X \times X}^{k-l}\} = A_1 \{(\pi_1^* \omega_X + \pi_2^* \omega_X)^{k-l}\}.$$

It follows that  $\{T_n\}$  is bounded from above by  $A_1^2$  times the class of

$$S := (\pi_1)_* [(\pi_1^* \pi^* \omega_Y + \pi_2^* \pi^* \omega_Y)^l \wedge (\pi_1^* \omega_X + \pi_2^* \omega_X)^{k-l} \wedge \pi_2^*(T)].$$

Observe that  $S$  is a linear combination of the forms  $\pi^*(\omega_Y^j) \wedge \omega_X^{p-j}$  with  $\max\{0, p-k+l\} \leq j \leq \min\{l, p\}$ . Moreover, the coefficient of  $\pi^*(\omega_Y^j) \wedge \omega_X^{p-j}$  in  $S$  is equal to the following constant function, i.e. closed  $(0,0)$ -current,

$$(\pi_1)_* [\pi_2^*(T) \wedge \pi_2^*(\omega_X^{k-l-p+j}) \wedge \pi_2^* \pi^*(\omega_Y^{l-j})] = (\pi_1)_* \pi_2^* [T \wedge \omega_X^{k-l-p+j} \wedge \pi^*(\omega_Y^{l-j})].$$

So, it is equal to the mass of the measure

$$T \wedge \omega_X^{k-l-p+j} \wedge \pi^*(\omega_Y^{l-j}).$$

Therefore, we have

$$S = \sum_{\max\{0, p-k+l\} \leq j \leq \min\{l, p\}} \alpha_j(T) \pi^*(\omega_Y^j) \wedge \omega_X^{p-j}.$$

The proposition follows.  $\square$

### 3 Proof of the main results

Let us start with the proof of Theorem 1.1. Although we follow closely the strategy for the main theorem in [8], our present exposition is simpler and more instructive thanks to the results of Section 2. For the sake of completeness and for the reader's convenience we give here the detailed proof.

First, we recall from Section 3 in [8] that the relative dynamical degrees are bi-meromorphic invariants. So, we can assume without loss of generality that  $\pi$  is a holomorphic map. Recall also that the relative dynamical degree  $d_p(f|\pi)$  of order  $p$ , with  $0 \leq p \leq k-l$ , is defined by

$$d_p(f|\pi) := \lim_{n \rightarrow \infty} [\lambda_p(f^n|\pi)]^{1/n},$$

where

$$\lambda_p(f^n|\pi) := \|(f^n)^*(\omega_X^p) \wedge \pi^*(\omega_Y^l)\| = \langle (f^n)^*(\omega_X^p) \wedge \pi^*(\omega_Y^l), \omega_X^{k-l-p} \rangle.$$

The reader will find in [8] the geometric interpretation of these degrees.

Our calculus involves the following auxiliary quantities. For  $n \geq 0$  and  $\max\{0, p-l\} \leq q \leq \min\{p, k-l\}$ , define

$$a_{q,p}(n) := \|(f^n)^*(\omega_X^p) \wedge \pi^*(\omega_Y^{l-p+q})\| = \langle (f^n)^*(\omega_X^p) \wedge \pi^*(\omega_Y^{l-p+q}), \omega_X^{k-l-q} \rangle.$$

Observe that

$$a_{p,p}(n) = \lambda_p(f^n|\pi). \quad (3)$$

Define also for  $0 \leq p \leq k$

$$b_p(n) := \sum_{\max\{0,p-l\} \leq q \leq \min\{p,k-l\}} a_{q,p}(n).$$

The following lemma shows that  $b_p(n)$  is equivalent to  $\lambda_p(f^n)$  when  $n$  goes to infinity.

**Lemma 3.1.** *The sequence  $b_p(n)^{1/n}$  converges to  $d_p(f)$ .*

*Proof.* Since  $\pi^*(\omega_Y^{l-p+q})$  is smooth on  $X$ , we have

$$a_{q,p}(n) = \langle (f^n)^*(\omega_X^p) \wedge \pi^*(\omega_Y^{l-p+q}), \omega_X^{k-l-q} \rangle \leq A \|(f^n)^*(\omega_X^p)\| = A\lambda_p(f^n)$$

for some constant  $A > 0$ . We deduce that

$$\limsup_{n \rightarrow \infty} b_p(n)^{1/n} \leq d_p(f).$$

So, in order to obtain the lemma, it is enough to check that  $\lambda_p(f^n) \leq Ab_p(n)$  for some constant  $A > 0$ .

Applying Proposition 2.5 to  $\omega_X^{k-p}$  gives

$$\{\omega_X^{k-p}\} \leq A \sum_{\max\{0,p-l\} \leq q \leq \min\{p,k-l\}} \{\pi^*(\omega_Y^{l-p+q})\} \cup \{\omega_X^{k-l-q}\}$$

for some constant  $A > 0$ . This, combined with the fact that  $(f^n)^*(\omega_X^p)$  is positive closed, implies that

$$\begin{aligned} \lambda_p(f^n) &= \langle (f^n)^*(\omega_X^p), \omega_X^{k-p} \rangle \\ &\leq A \sum_{\max\{0,p-l\} \leq q \leq \min\{p,k-l\}} \langle (f^n)^*(\omega_X^p), \pi^*(\omega_Y^{l-p+q}) \wedge \omega_X^{k-l-q} \rangle \\ &\leq A \sum_{\max\{0,p-l\} \leq q \leq \min\{p,k-l\}} a_{q,p}(n). \end{aligned}$$

The last sum is equal to  $b_p(n)$ . The lemma follows.  $\square$

For  $n \geq 0$  and  $0 \leq p \leq l$ , define

$$c_p(n) := \lambda_p(g^n) = \|(g^n)^*(\omega_Y^p)\| = \langle (g^n)^*(\omega_Y^p), \omega_Y^{l-p} \rangle.$$

We have the following lemmas.

**Lemma 3.2.** *There is a constant  $A > 0$  such that*

$$\left\langle (f^n)^*(\pi^*\omega_Y^{p-q} \wedge \omega_X^q), \pi^*(\omega_Y^{l-p+p_0}) \wedge \omega_X^{k-l-p_0} \right\rangle \leq Aa_{p_0,q}(n)c_{p-q}(n)$$

for  $0 \leq p_0 \leq k-l$ ,  $p_0 \leq p \leq l+p_0$ ,  $p_0 \leq q \leq p$  and  $n \geq 0$ . Moreover, the above integral vanishes when  $q < p_0$ .

*Proof.* We prove the first assertion. Observe that

$$(f^n)^*(\pi^*\omega_Y^{p-q} \wedge \omega_X^q) = (f^n)^*(\pi^*\omega_Y^{p-q}) \overset{\circ}{\wedge} (f^n)^*(\omega_X^q).$$

Hence, the left hand side of the inequality in the lemma is equal to

$$\left\langle (f^n)^*\pi^*\omega_Y^{p-q} \overset{\circ}{\wedge} (f^n)^*(\omega_X^q), \pi^*(\omega_Y^{l-p+p_0}) \wedge \omega_X^{k-l-p_0} \right\rangle.$$

Define

$$T := (f^n)^*\pi^*\omega_Y^{p-q} \wedge \pi^*(\omega_Y^{l-p+p_0}) \quad \text{and} \quad S := (f^n)^*(\omega_X^q) \wedge \omega_X^{k-l-p_0}.$$

These currents are of bidegree  $(l-q+p_0, l-q+p_0)$  and  $(k-l+q-p_0, k-l+q-p_0)$  respectively. They are almost-smooth and hence have no mass on proper analytic subsets. The left hand side of the inequality in the lemma is equal to the mass of the measure  $T \overset{\circ}{\wedge} S$ . Since  $\pi \circ f^n = g^n \circ \pi$ , we have

$$T = (f^n)^*\pi^*(\omega_Y^{p-q}) \wedge \pi^*(\omega_Y^{l-p+p_0}) = \pi^\bullet(g^n)^*(\omega_Y^{p-q}) \wedge \pi^*(\omega_Y^{l-p+p_0}).$$

By Proposition 2.1, for every fixed  $n$ , there exist smooth positive closed forms  $\beta_j$  of bidegree  $(p-q, p-q)$  on  $Y$  so that

- $\|\beta_j\| \leq A\|(g^n)^*(\omega_Y^{p-q})\| = Ac_{p-q}(n)$  for all  $j$ ;
- $\lim_{j \rightarrow \infty} \beta_j \geq (g^n)^*(\omega_Y^{p-q})$ ,

where  $A > 0$  is a constant that depends only on  $Y$ . Then, using (1), we deduce from the above discussion that

$$T \leq \lim_{j \rightarrow \infty} \pi^*(\beta_j) \wedge \pi^*(\omega_Y^{l-p+p_0}) = \lim_{j \rightarrow \infty} \pi^*(\beta_j \wedge \omega_Y^{l-p+p_0}).$$

Hence, since  $T$  and  $S$  are almost-smooth, we obtain

$$\|T \overset{\circ}{\wedge} S\| \leq \lim_{j \rightarrow \infty} \left\langle \pi^*(\beta_j \wedge \omega_Y^{l-p+p_0}), S \right\rangle.$$

Since  $\pi^*(\beta_j \wedge \omega_Y^{l-p+p_0})$  are smooth, the right hand side of the above inequality increases when we replace  $\beta_j$  by any closed smooth form having a larger cohomology class. Consequently,

$$\lim_{j \rightarrow \infty} \left\langle \pi^*(\beta_j \wedge \omega_Y^{l-p+p_0}), S \right\rangle \lesssim c_{p-q}(n) \|\pi^*(\omega_Y^{l-q+p_0}) \wedge S\| = c_{p-q}(n)a_{p_0,q}(n).$$

This completes the proof of the first assertion.

For the second assertion, when  $q < p_0$  the form  $\beta_j \wedge \omega_Y^{l-p+p_0}$  has bidegree  $(l-q+p_0, l-q+p_0)$  which is bigger than  $(l, l)$ , thus must be 0 since  $Y$  has dimension  $l$ . It follows that  $T = 0$  and the integral in the lemma is 0 as well.  $\square$

**Lemma 3.3.** *There exists a constant  $A > 0$  such that for all  $0 \leq p_0 \leq k - l$ ,  $p_0 \leq p \leq l + p_0$  and all  $n, r \geq 1$*

$$a_{p_0,p}(nr) \leq A^r \sum \prod_{s=1}^r a_{p_{s-1},p_s}(n) c_{p-p_s}(n),$$

where the sum is taken over  $(p_1, \dots, p_r)$  with  $p_0 \leq p_1 \leq p_2 \leq \dots \leq p_r \leq p$  and  $p_{r-1} \leq k - l$ .

*Proof.* We proceed by induction on  $r$ . Clearly, the lemma is true for  $r = 1$ . Suppose the lemma for  $r$ , we need to prove it for  $r + 1$ . In what follows,  $\lesssim$  denotes an inequality up to a multiplicative constant which depends only on the geometry of  $X$  and  $Y$ .

Define  $T^{(r)} := (f^{nr})^*(\omega^p)$ . This is an almost-smooth current on  $X$ . Therefore, we have

$$T^{(r+1)} = (f^n)^\bullet(T^{(r)}).$$

By Proposition 2.5 applied to  $T^{(r)}$ , we can find smooth positive closed  $(p, p)$ -forms  $T_i^{(r)}$  converging weakly to a positive closed current  $\tilde{T}^{(r)} \geq T^{(r)}$  such that

$$\alpha_{p-q}(T_i^{(r)}) \lesssim \alpha_{p-q}(T^{(r)}) \lesssim a_{q,p}(nr)$$

for  $\max\{0, p - l\} \leq q \leq \min\{p, k - l\}$ . Then, using again that proposition, we deduce that

$$\{T_i^{(r)}\} \lesssim \sum_{\max\{0, p-l\} \leq q \leq \min\{p, k-l\}} a_{q,p}(nr) \{\pi^*(\omega_Y^{p-q})\} \smile \{\omega_X^q\}.$$

Finally, we obtain from the above discussion and Lemma 3.2 that

$$\begin{aligned} a_{p_0,p}(n(r+1)) &= \langle T^{(r+1)}, \pi^*(\omega_Y^{l-p+p_0}) \wedge \omega_X^{k-l-p_0} \rangle \\ &\leq \liminf_{i \rightarrow \infty} \langle (f^n)^*(T_i^{(r)}), \pi^*(\omega_Y^{l-p+p_0}) \wedge \omega_X^{k-l-p_0} \rangle \\ &\lesssim \sum_{\max\{0, p-l\} \leq q \leq \min\{p, k-l\}} a_{q,p}(nr) \langle (f^n)^*(\pi^*(\omega_Y^{p-q}) \wedge \omega_X^q), \pi^*(\omega_Y^{l-p+p_0}) \wedge \omega_X^{k-l-p_0} \rangle \\ &\lesssim \sum_{p_0 \leq q \leq \min\{p, k-l\}} a_{q,p}(nr) a_{p_0,q}(n) c_{p-q}(n). \end{aligned}$$

These estimates together with the induction hypothesis imply the result.  $\square$

**Proposition 3.4.** *We have*

$$d_p(f) \geq \max_{\max\{0, p-k+l\} \leq j \leq \min\{p, l\}} d_j(g) d_{p-j}(f|\pi)$$

for  $0 \leq p \leq k$ .

*Proof.* Since  $\pi^*(\omega_Y^j) \wedge \omega_X^{p-j}$  is a smooth  $(p, p)$ -form, we have

$$\|(f^n)^*(\pi^*(\omega_Y^j) \wedge \omega_X^{p-j})\| \lesssim \lambda_p(f^n).$$

So, by definition of dynamical degrees and equality (3), it is enough to bound  $\|(f^n)^*(\pi^*(\omega_Y^j) \wedge \omega_X^{p-j})\|$  from below by a constant times  $\lambda_j(g^n)a_{p-j,p-j}(n)$ .

Using the identity  $\pi \circ f^n = g^n \circ \pi$  and that  $\pi^*(\omega_Y^{l-j}) \wedge \omega_X^{k-l-p+j}$  is smooth, we obtain

$$\begin{aligned} & \|(f^n)^*(\pi^*\omega_Y^j \wedge \omega_X^{p-j})\| \\ & \geq \langle (f^n)^*(\pi^*\omega_Y^j \wedge \omega_X^{p-j}), \pi^*(\omega_Y^{l-j}) \wedge \omega_X^{k-l-p+j} \rangle \\ & = \langle (f^n)^*\pi^*(\omega_Y^j) \overset{\circ}{\wedge} (f^n)^*(\omega_X^{p-j}), \pi^*(\omega_Y^{l-j}) \wedge \omega_X^{k-l-p+j} \rangle \\ & = \|(f^n)^*\pi^*(\omega_Y^j) \wedge \pi^*(\omega_Y^{l-j}) \overset{\circ}{\wedge} (f^n)^*(\omega_X^{p-j}) \wedge \omega_X^{k-l-p+j}\| \\ & = \|\pi^\bullet[(g^n)^*(\omega_Y^j) \wedge \omega_Y^{l-j}] \overset{\circ}{\wedge} (f^n)^*(\omega_X^{p-j}) \wedge \omega_X^{k-l-p+j}\|. \end{aligned}$$

Observe that  $(g^n)^*(\omega_Y^j) \wedge \omega_Y^{l-j}$  is a positive measure of mass  $\lambda_j(g^n)$ . Using a simple argument on cohomology as in Lemma 3.2 in [8], we show that the last expression is equal to  $\lambda_j(g^n)$  times the mass of the restriction of  $(f^n)^*(\omega_X^{p-j}) \wedge \omega_X^{k-l-p+j}$  to a generic fiber of  $\pi$ . Therefore, it is also equal to

$$\lambda_j(g^n) \langle \pi^*(\omega_Y^l), (f^n)^*(\omega_X^{p-j}) \wedge \omega_X^{k-l-p+j} \rangle = \lambda_j(g^n)a_{p-j,p-j}(n),$$

where for simplicity we normalize  $\omega_Y$  so that  $\omega_Y^l$  is a probability measure. This completes the proof of the proposition.  $\square$

**Proof of Theorem 1.1.** By Proposition 3.4, we only need to show that

$$d_p(f) \leq \max_{\max\{0, p-k+l\} \leq j \leq \min\{p, l\}} d_j(g)d_{p-j}(f|\pi)$$

for  $0 \leq p \leq k$ . To do this we argue exactly as in the proof of Proposition 4.6 in [8] using identity (3), Lemma 3.1 and Lemma 3.3.  $\square$

In the rest of the paper we prove the corollaries of Theorem 1.1.

**Proof of Corollary 1.2.** Using Theorem 1.1, we proceed as in the proof of Corollary 1.3 in [8].  $\square$

**Proof of Corollary 1.3.** Using Theorem 1.1, we argue as in the proof of Corollary 1.4 in [8].  $\square$

We recall here briefly the definition of the Albanese fibration of a compact Kähler manifold  $X$ . Let  $H^0(X, \Omega_X)$  be the complex vector space of all holomorphic 1-forms on  $X$ . Since  $X$  is compact Kähler, these forms are closed. Therefore, to any closed path  $\gamma$  we associate the linear form

$$H^0(X, \Omega_X) \ni \varphi \mapsto \int_\gamma \varphi$$



which depends only on the homology class of  $[\gamma] \in H_1(X, \mathbb{Z})$ . This correspondence identifies the component without torsion of  $H_1(X, \mathbb{Z})$  with a co-compact lattice  $\Gamma$  of the dual space  $H^0(X, \Omega_X)^*$ . The Albanese variety  $\text{Alb}(X)$  of  $X$  is, by definition, the complex torus  $H^0(X, \Omega_X)^*/\Gamma$ .

Fix a base point  $x \in X$ . Let  $y \in X$  and  $\varphi \in H^0(X, \Omega_X)$ . Then, for different paths  $\gamma$  connecting  $x$  to  $y$ , the corresponding values of  $\int_\gamma \varphi$  are always equal modulo the values of  $\int_\delta \varphi$  for some closed path  $\delta$ . Consequently, we obtain a holomorphic map  $\text{alb} : X \rightarrow \text{Alb}(X)$  defined by

$$\text{alb}(y) := \int_x^y \varphi, \quad \varphi \in H^0(X, \Omega_X),$$

where the integration is taken over an arbitrary path  $\gamma$  connecting  $x$  to  $y$ . This is the Albanese map of  $X$ .

**Proof of Corollary 1.4.** Let  $Y := \text{alb}(X)$  be the image of the Albanese map. If  $\varphi$  is a holomorphic 1-form, then  $f^*(\varphi)$  is a holomorphic 1-form outside an analytic set of codimension  $\geq 2$ . By Hartogs' theorem, this form extends to a holomorphic 1-form on  $X$ . Therefore,  $f$  induces a linear operator  $f^*$  from  $H^0(X, \Omega_X)$  to itself.

This operator induces a dominant meromorphic map  $g$  on  $Y$  such that  $f$  is alb-semi-conjugate to  $g$ . By Corollary 1.2, the assumption on  $f$  implies that  $g$  also has distinct consecutive dynamical degrees (since dynamical degrees are bi-meromorphic invariants, we can desingularize  $Y$  if necessary). By Corollary 1.3, the Kodaira dimension  $\kappa_Y$  of  $Y$  satisfies  $\kappa_Y \leq 0$ .

On the other hand, by Corollary 10.6 in Ueno's book [22],  $\kappa_Y \geq 0$ . Hence,  $\kappa_Y = 0$ . But by this corollary again, we have  $Y = \text{alb}(X)$ .  $\square$

**Proof of Corollary 1.5.** Let  $c_X : X \rightarrow C(X)$  be the core fibration constructed by Campana in [4]. By the proof of Theorem 6.1 in [1],  $C(X)$  is a projective variety. Moreover, there exists a bi-meromorphic map  $c_f : C(X) \rightarrow C(X)$  such that  $c_X \circ f = c_f \circ c_X$  and that  $c_f^n = \text{id}$  for some  $n \geq 1$ .

A priori,  $C(X)$  may be singular, but we can use a blow-up and assume that  $C(X)$  is a smooth projective manifold. Clearly,  $d_j(c_f) = 1$  for  $0 \leq j \leq \dim C(X)$ . By Corollary 1.2, it follows from the assumption on  $f$  that  $\dim C(X) = 0$ . Thus,  $c_X$  is a constant map. Since Theorem 3.3 in [4] says that the generic fibers of  $c_X$  are special, so is  $X$ .  $\square$

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